Cut-out sets, fractal voids and cosmic structure

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Abstract

"Cut-out sets" are fractals that can be obtained by removing a sequence of disjoint regions from an initial region of d-dimensional euclidean space. Conversely, a description of some fractals in terms of their void complementary set is possible. The essential property of a sequence of fractal voids is that their sizes decrease as a power law, that is, they follow Zipf's law. We prove the relation between the box dimension of the fractal set (in $d \leq 3$) and the exponent of the Zipf law for *convex* voids; namely, if the Zipf law exponent e is such that $1 < e < d/(d-1)$ and, in addition, we forbid the appearance of *degenerate* void shapes, we prove that the corresponding cut-out set has box dimension d/e ($d-1 < d/e < d$). We explore the application of this result to the large scale distribution of matter in cosmology, in connection with "cosmic foam" models.

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1 Introduction

Some fractals can be obtained by removing an infinite sequence of disjoint regions from an initial set. Mandelbrot [1] introduced the concept of fractal holes (under the greek word *tremas*) and showed that the distribution of one-dimensional holes (gaps) follows a simple power law; namely, the number of gaps of length U greater than u is $N(U > u) \propto u^{-D}$, where D is the fractal dimension. Falconer [2] has studied the fractal properties of "cut-out sets" in terms of box dimensions. In particular, he has proved that fractal sets in one dimension, that is, fractal subsets of $\mathbb{R}$, have box dimensions that depend on the size of the complementary intervals and not on their arrangement. To be precise, the fractal E, with a sequence of void intervals a_k ($k = 1, 2, \ldots$) has

$$\dim_B E = -1/\lim_{k\to\infty}(\log a_k/\log k)$$

if and only if the limit exists. Roughly speaking, this theorem says that fractality in $\mathbb{R}$ is related with Zipf's power law [7] for the rank-ordering of void intervals (with exponent $1/\dim_B E$). In turn, Zipf's law is equivalent to the law $N(U > u) \propto u^{-D}$ if we identify D with $\dim_B E$.

Mandelbrot and Falconer have also considered the generalization to higher dimension, that is, to fractal subsets of $\mathbb{R}^d$. However, a void in a subset of $\mathbb{R}^d$ is not only defined by its size, so the problem is more complex. One can prescribe a definite shape of voids, for example, discs in $d = 2$, so that the size determines the void (up to its position). Then it still holds that the box dimension of a cut-out set depends only on the sequence of sizes of the complementary intervals and does not depend on their arrangement. However, whereas void regions can be freely rearranged in $d = 1$, in $d > 1$ there are constraints. Examples of two-dimensional cut-out sets are the Apollonian packings of disks [1, 2].

In addition, while in $d = 1$ there is no restriction on the fractal (Hausdorff-Besicovitch or box) dimension of cut-out sets, except that it be between zero and one, in $d > 1$, cut-out sets must have *topological codimension* one, so they are formed by curves in two dimensions or surfaces in three dimensions (etc.) which enclose the voids. Therefore, their fractal dimension must be larger than $d - 1$.

On the other hand, it is possible to define a sequence of void regions in a finite approximations to a fractal set by algorithmic methods that exploit the recursive structure of fractals [3, 4]. These algorithms are called *void-finders*. The question is to determine the relation between the voids thus found and the actual fractal set; for example, to discern in what sense (if any) a general fractal can be considered a cut-out set of the voids thus found.

The motivation to construct void-finders has arisen in cosmology, but the interest of fractals for cosmology is older. It rests on the fact that the large scale distribution of matter is arguably scale invariant (within some limits). The fractal properties of the distribution of galaxies have been studied for many years (recent references are in [5]). The existence of large voids in the distribution of galaxies, as a counterpart of galaxy clustering, is an observational fact. The distribution and properties of those voids have become a subject of systematic study [6]. The understanding of voids as a fractal feature of the distribution of galaxies motivated us to study general properties of fractal voids [3, 4]. In particular, the already compiled catalogues of voids in the distribution of galaxies are suitable for standard rank-ordering techniques [7, 8] in which voids are ranked by size. In Ref. [3] we defined voids as having constant shape: discs or squares, in two dimensions (the disc case is connected with Apollonian packings). This definition is adequate for mathematical treatment, but we found that it is not satisfactory for analysing a general fractal: often equal-shape voids of similar size touch each other and visually it seems that they should be merged into a unique void of *irregular* shape. This suggested us to improve the definition of void by considering voids of arbitrary shape. In particular, in Ref. [4], we devised a void-finding algorithm based on discrete geometry methods, namely, Delaunay and Voronoi tessellations. It produces a sequence of voids of polygonal but otherwise arbitrary shape (in $d = 2$). The rank-ordering of the voids of suitable examples of fractal point sets in one and two dimensions showed that Zipf's power-law holds; but we did not attempt to

provide any mathematical proof of this.

So the purpose of this work is: (i) to extend the notion of cut-out set in $d > 1$ to void regions of non-constant shape and to establish the relation of the box dimension of the set with the Zipf law for voids, following Falconer's methods [2]; (ii) to see in what sense and to what extent a general fractal can be considered a cut-out set of the voids found by algorithmic methods; (iii) finally, to indicate how to apply our conclusions to the large scale distribution of matter in cosmology.

To achieve the first goal (i), we will need to introduce concepts of integral geometry [9]. These concepts, in particular, the Minkowski functionals, have already been applied to fractal distributions by Mecke [10]. His point of view, based on clusters, can be considered dual to the point of view that we adopt here, based on voids.

2 Cut-out sets in more than one dimension

Cut-out sets are obtained by removing an infinite sequence of disjoint regions from an initial set. In one dimension, a cut-out set is a subset of $\mathbb{R}$ obtained by removing an infinite sequence of disjoint open intervals from an initial closed interval such that the sum of the lengths of the removed intervals converges to the total length of the initial interval. A trivial example is the Cantor set. Let the fractal $E \in \mathbb{R}$ be the result of removing from the closed interval A a sequence of open disjoint intervals $\{A_k\}_{k=1}^{\infty}$ such that the length of A_k is a_k and $\sum_{k=1}^{\infty} a_k$ equals the length of A. It is not difficult to see that the set obtained by removing a finite number of intervals of decreasing length is related with a neighbourhood of E and so the pattern of sizes of removed intervals is related with the behaviour of the r-neighbourhood of E as $r \to 0$. If this behaviour is a power law, it defines the Minkowski-Bouligand dimension, which is equivalent to the box dimension [1, 11]. A careful analysis then shows that the corresponding condition on the lengths of removed intervals is that they follow a power law of their rank, that is, that they follow Zipf's law [7]. To be precise, the fractal $E = A - \cup_{k=1}^{\infty} A_k$, with the decreasing sequence of lengths $\{a_k\}_{k=1}^{\infty}$, has box dimension

$$\dim_B E = -1/ \lim_{k\to\infty} (\log a_k / \log k)$$

if and only if the limit exists.

In more than one dimension, a cut-out set is a subset of $\mathbb{R}^d$ obtained by removing an infinite sequence of disjoint connected open regions $\{A_k\}_{k=1}^{\infty}$ from an initial compact region A, which is natural to choose convex [2]. We need to restrict the possible shapes of the A_k. The simplest option is to prescribe a definite shape for them. Before we analyse the problem in detail, let us consider a particular type of fractals with voids of constant shape.

2.1 Cantor-like fractals and merging criteria

For Cantor-like fractals, which are strictly self similar, Zipf's law for voids is a consequence of their construction [3]. A (deterministic) Cantor set is constructed with a generator

characterized by two numbers, namely, the scaling factor r and the number of pieces N to remain, and by the arrangement of these remaining pieces. In one dimension, we convene that if the arrangement of the removed $1/r - N$ *open* intervals is such that two or more are adjacent then one also removes the isolated points between them (*merging* criterium). This criterium leads to characterize the fractal generator by one more number: its number of voids (gaps) $m \leq 1/r - N$ [3]. If we do not apply the merging criterium, the resulting countable set of isolated points does not contribute to the Hausdorff-Besicovitch dimension of the Cantor-like set but may contribute to its box dimension [11].

In d dimensions, the merging criterium generalizes to the removal of the $d-1$-dimensional boundaries between adjacent open d-cubes. However, a complication arises: it is also possible that some open d-cubes touch the boundary of the initial closed d-cube. If we also convene to remove the corresponding $d-1$-dimensional boundaries, then, when we iterate the generator, we merge voids of different levels. Actually, all the void regions merge and form one connected void region. We may not allow this by preserving in the generator the boundary of the initial closed d-cube. This boundary has topological codimension one, so the box and Hausdorff-Besicovitch dimensions of the resulting Cantor-like set are larger than $d-1$. This holds even when the expected value of the box and Hausdorff-Besicovitch dimensions, $-\log N/\log r$, is smaller than $d-1$, so then the boundary preserving criterium alters them significantly. Even if the expected Hausdorff-Besicovitch dimension is larger than $d-1$, so that it may not be altered by this criterium, the structure of the resulting Cantor-like set may change significantly.

For example, consider the two-dimensional middle third Cantor set, constructed as the cartesian product of two middle third Cantor sets and with dimension $2\log 2/\log 3 > 1$; it has zero topological dimension, that is, it is a set of points rather than lines. Its construction with the two-dimensional generator consisting of removing the appropriate pattern of five open squares from the unit closed square with the boundary preserving criterium, makes it a set of lines (with the same box and Hausdorff-Besicovitch dimensions as the point set). This happens irrespective of any criterium for merging accross inter-cube boundaries: if no merging is prescribed, then there are additional lines.

Of course, if the generator does not include any open d-cube that touches the boundary of the initial closed d-cube, cubes removed in different iterations cannot merge. This condition ensures that the fractal set has topological codimension one. Typical fractals generated in this form are the Sierpinski carpets [1]. Assuming inter-cube merging, it is convenient to introduce the number m of voids of the generator, such that $1 \leq m \leq 1/r^d - N$. The Hausdorff-Besicovitch dimension $D = -\log N/\log r > d-1$ is independent of m and the sizes a_k follow a power law with exponent $-d/D$ (on average) [3, 4].

Note that the inter-cube merging criterium implies that voids in the generator may not have a common shape but a finite number of shapes instead. However, this finite number of void shapes is conserved in the iterations, which only change their size.

2.2 Voids in general fractals and cut-out sets

We have seen in the preceding examples of Cantor-like fractals that a precise definition of voids demands them to have a precise boundary and, therefore, the corresponding fractal set has topological codimension one, since it contains the boundaries which have themselves codimension one. Fractals of topological codimension one are curves in $d = 2$, surfaces in $d = 3$, etc. One may wonder if a codimension one set has an associated set of voids that make it equivalent to a cut-out set, and if those voids fulfill Zipf's law. Since we take the initial compact region A of a cut-out set to be convex, we may consider the convex hull of the fractal and the complementary set of the fractal with respect to it. If it is formed by an infinite set of connected regions, this is the natural set of voids. For example, this approach works for the von Koch curve, whose set of voids can be shown to fulfill Zipf's law by relying on its self-similarity. However, these voids have themselves fractal boundary and, therefore, are not sufficiently simple for our purposes.

A different approach is algorithmic, namely, to define fractal voids as the regions found by void-finders. These regions may have constant shape, being the obvious shapes a square or a disc [3]. Or we may allow arbitrary shapes; in particular, we have devised a new void-finder, based on discrete geometry methods (Delaunay and Voronoi tessellations), that produces a sequence of voids of arbitrary polygonal or polyhedral shape, respectively, in $d = 2$ or in $d = 3$ [4].

The problem with the constant shape voids found in an arbitrary fractal is that they may not fill its real voids of variable shape, if they exist. For example, let us assume that a fractal in $d = 2$ has a square void that we are trying to fill with discs. This filling will form the Apollonian packing of the square, which leaves out a fractal region of dimension about 1.31 [2]. Then it may happen that this dimension is larger than the dimension of the original fractal with the square void. In contrast, arbitrary polygonal (or polyhedral) shapes are adaptable. Therefore, if a fractal has topological codimension one and so it has well-defined voids, appropriate polygonal (or polyhedral) shapes will reproduce them with sufficient approximation. However, this approach still allows for too general void shapes because the voids may have fractal boundary and diverging perimeter length (in $d = 2$, or diverging surface area in $d = 3$).

Arbitrary polygonal (or polyhedral) void shapes seem a suitable starting point but they are still too general. We would like them to have a bounded perimeter length (or surface area) for a given *diameter* (we define diameter as the greatest distance apart of pairs of points [11]). A nice way of implementing this condition without being too restrictive, that is, of preventing the boundary of voids from becoming wrinkled, is to require them to be convex. So we define henceforth a cut-out set as a subset of $\mathbb{R}^d$ obtained by removing an infinite sequence of disjoint open convex d-polyhedral regions $\{A_k\}_{k=1}^{\infty}$ from an initial compact convex region A. In the end, the restriction to d-polyhedra proves to be superfluous and we may consider general convex voids. However, from a computational point of view, that is, when finding voids in a finite sets of points, these voids must be polyhedra. In the next section, we generalize the results of Falconer regarding the box dimension of fractals resulting from cutting out discs [2] to these more general cut-out sets, restricting ourselves

to $d = 2$ as well. Further generalization to $d = 3$ is achieved in the following section.

2.3 Two-dimensional cut-out sets with convex polygonal voids

Let A be a plane compact convex region of perimeter p and let $\{A_k\}_{k=1}^{\infty}$ be a sequence of disjoint open convex polygonal regions, such that A_k has diameter δ_k, perimeter p_k, and area a_k. Contrary to the case of discs (or other constant shapes), these three quantities are independent. So there is no obvious order of the A_k. Let $\{A_k\}_{k=1}^{\infty}$ be such that the total area $\sum_{k=1}^{\infty} a_k$ is equal to the area of A. Then the set $E = A - \cup_{k=1}^{\infty} A_k$ has zero area. To calculate its box dimension, we can use the Minkowski-Bouligand dimension [1, 11], which is equivalent to the box dimension [11].

The Minkowski-Bouligand dimension expresses the power-law behaviour of the r-neighbourhood of E as $r \to 0$. To apply it to the cut-out set E, one needs to distinguish voids that are included in the r-neighbourhood of E from voids that are not included. In any one of the latter, the r-neighbourhood of E is a band of width r that leaves an empty part. We can try to estimate the areas of these bands, in order to obtain the area of the r-neighbourhood of E. Before doing so, let us recall Falconer's formula for the area of the r-neighbourhood of E in the case of disc-shaped voids.

2.3.1 Area of the r-neighbourhood of E with disc-shaped voids

Falconer's formula [2] for the area of the r-neighbourhood of E in the case of disc-shaped voids reads:

$$V(r) = (p\,r + \pi r^2) + \sum_{i=1}^{k} \pi \left(r_i^2 - (r_i - r)^2\right) + \sum_{i=k+1}^{\infty} \pi\, r_i^2 \tag{1}$$

$$= p\,r + 2\pi r \sum_{i=1}^{k} r_i + \pi \sum_{i=k+1}^{\infty} r_i^2 + \pi\, r^2(1-k), \tag{2}$$

where k is such that r is between the disc radii of indices k and $k+1$ ($r_{k+1} \le r \le r_k$). The first term of Eq. (1) is the area of a band surrounding A, the second term is the area of the annuli of width r inside the discs A_i for $1 \le i \le k$, and the third term the area of the discs A_i for $i \ge k+1$. In the second expression, Eq. (2), the second term, which is the area of the annuli of width r except for $-k\pi r^2$ (added to the last term), represents the sum of the perimeters of the discs A_i for $1 \le i \le k$ multiplied by r.

Falconer assumes that the disc radii and, therefore, areas fulfill Zipf's law, with a concrete formulation, namely, $r_k \asymp k^{-\alpha}$, in terms of the relation $\asymp$. The relation $a_k \asymp b_k$ between two sequences $\{a_k\}$ and $\{b_k\}$ means that there are two constants $c_1, c_2 > 0$ such that $c_1 \le a_k/b_k \le c_2$ for all k (this concept also applies to functions, having the relation $f(x) \asymp g(x)$ analogous meaning). Falconer further assumes that $1/2 < \alpha < 1$. The inequality $\alpha > 1/2$ ensures that the series of disc areas converges. The role of the inequality $\alpha < 1$ is more subtle: it implies that the series of disc perimeters diverges. So, for

$r_{k+1} \leq r \leq r_k$, we have

$$V(r) \asymp r + r\sum_{i=1}^{k} i^{-\alpha} + \sum_{i=k+1}^{\infty} i^{-2\alpha} - r^2 k \asymp r\, k^{1-\alpha} + k^{1-2\alpha} \asymp r^{2-1/\alpha}, \tag{3}$$

and

$$\dim_B E = 2 - \lim_{r\to 0} \frac{\log V(r)}{\log r} = \frac{1}{\alpha} > 1.$$

If $\alpha > 1$ the series of perimeters converges and the second term in Eq. (2) behaves like r, dominating over the terms that behave like $r^{2-1/\alpha}$. This implies that $\dim_B E = 1$, instead of $\dim_B E = 1/\alpha < 1$. We encounter again that a cut-out set must have dimension larger than $d-1$, since this is the topological dimension of the boundaries of voids.

2.3.2 Area of the r-neighbourhood of E with triangular voids

Before proceeding to analysing the case of general convex polygonal voids, let us analyse the simpler case of triangles (which are necessarily convex).

Let us consider voids that are not included in the r-neighbourhood of E. In any of these voids, the r-neighbourhood of E forms a band of width r that leaves an empty part, which we can calculate. To be precise, the r-neighbourhood of the boundary of a triangle is a band of width r so the region uncovered is another triangle with the same angles and, therefore, similar. The similarity ratio can be determined employing some elementary geometry. The vertices of the similar triangles for various r are placed along the angle bisectors and so the triangles themselves are homothetical with respect to the common centre of their inscribed circles. The area of any of these triangles is given in terms of its perimeter p and the radius of its inscribed circle ρ by $a = p\rho/2$. If ρ refers to the original triangle and ρ' to another, the similarity ratio is ρ'/ρ. So the area of the latter is $a' = a(\rho'/\rho)^2$ and $a - a' = a[1-(\rho'/\rho)^2]$. Given that $r = \rho - \rho'$, we obtain for the area of the band of width r

$$a - a' = p\, r - \frac{p\, r^2}{2\rho} = p\, r - \frac{p^2 r^2}{4a}. \tag{4}$$

In this formula, the factor $p^2/(4a)$ of r^2 only depends on the angles and is a measure of shape. The value of a' decreases as r increases and vanishes for the maximal $r = \rho$.

The preceding results show us the appropriate version of formulas (1) and (2) for the area of the r-neighbourhood of E:

$$V(r) = (p\, r + \pi r^2) + \sum_{i=1}^{k} \left(p_i r - \frac{p_i^2 r^2}{4a_i} \right) + \sum_{i=k+1}^{\infty} a_i \tag{5}$$

$$= p\, r + r\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{\infty} a_i + r^2 \left(\pi - \sum_{i=1}^{k} \frac{p_i^2}{4a_i} \right), \tag{6}$$

where k is such that $\rho_{k+1} \leq r \leq \rho_k$. To proceed like in the case of discs, we would like to have

$$\rho_k \asymp k^{-\alpha}, \ p_k \asymp \rho_k, \ a_k \asymp \rho_k^2 .$$

So we must have that $a_k \asymp p_k^2$, that is, that the perimeter to area ratio $p^2/(4a)$ is bounded above and below. According to the basic *isoperimetric inequality*, this quantity has an absolute lower bound of π, reached by a disc (for triangles the lower bound is larger, namely, $3\sqrt{3}$, reached by the equilateral triangle) [9]. The upper bound must be explicitly imposed. Then, from $\rho_k \asymp k^{-\alpha}$ and $a_k \asymp p_k^2$ we deduce $p_k \asymp k^{-\alpha}$. Therefore, a proof analogous to Falconer's proof for disc voids leads to the same result, namely, $\dim_B E = 1/\alpha$ $(1/2 < \alpha < 1)$.

An upper bound to the perimeter to area ratio $p^2/(4a)$ is an intuitively reasonable requirement. This ratio only depends on the angles of the triangle and an upper bound to it is equivalent to a lower bound to them. In other words, we are excluding "spiky" triangles, which are nearly one-dimensional and can be packed in a small area without reducing their diameter.

2.3.3 Area of the r-neighbourhood of E with convex polygonal voids

In this case, in a void convex polygon not included in the r-neighbourhood of E, the r-neighbourhood also forms a band of width r that leaves an empty part, but its area is harder to calculate. Therefore, instead of attempting to derive an equation analogous to Eq. (6), we look for independent bounds to $V(r)$. A lower bound is certainly provided by the sum of the areas of polygons fully included in the r-neighbourhood. An upper bound to $V(r)$ requires us to estimate the area of bands of with r inside larger polygons and will be given by an expression simpler than the right-hand side of Eq. (6).

To find a precise expression of the lower bound to $V(r)$, we need a criterium to determine when a void polygon is fully included in the r-neighbourhood of E. A simple criterium is given by the diameter δ of the polygons: if $r \geq \delta$, then the void polygons of diameter equal or smaller than δ are certainly covered. So, if we order the void polygons by their diameter and k is such that $\delta_{k+1} \leq r \leq \delta_k$,

$$\sum_{i=k+1}^{\infty} a_i \leq V(r). \tag{7}$$

To find a precise expression of the upper bound to $V(r)$, we need a criterium to determine when a void polygon is *not* fully included in the r-neighbourhood of E. Take a particular polygon A_i, with area a_i and perimeter p_i. Given that the area of a band of width r inside A_i and around its boundary is smaller than $p_i\, r$, a sufficient condition is that $r \leq a/p$. Therefore, an upper bound to $V(r)$ is

$$V(r) \leq pr + r \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{\infty} a_i + \pi r^2 , \tag{8}$$

where the void polygons are ordered by their value of a/p and k is such that

$$\frac{a_{k+1}}{p_{k+1}} \leq r \leq \frac{a_k}{p_k}\,.$$

We again need the relation $a_k \asymp p_k^2$ to relate the second and third terms of inequality (8). This condition implies a lower bound to the angles of polygons, like for triangles. However, the condition is now stronger: it is possible to have "spiky" convex polygons (with large diameter to area ratio) with non-small angles; a simple example is the rectangle, with no upper bound to the ratio $p^2/(4a)$.

Then, let us impose the conditions

$$p_k \asymp k^{-\alpha},\; a_k \asymp k^{-2\alpha}\,,$$

like we did for triangles. We have, for $a_{k+1}/p_{k+1} \leq r \leq a_k/p_k$,

$$V(r) \leq c\left(r\sum_{i=1}^{k} i^{-\alpha} + \sum_{i=k+1}^{\infty} i^{-2\alpha}\right) + \pi r^2 \leq c'(r\,k^{1-\alpha} + k^{1-2\alpha}) + \pi r^2 \leq c'' r^{2-1/\alpha}, \tag{9}$$

for some positive numbers c, c', c'', and $r < 1$. On the other hand, we can apply a similar procedure to the lower bound. To do so, we first relate the diameter of a convex polygon with its perimeter. We note that there are both lower and upper bounds to their ratio: $2\delta < p$ and $p \leq \pi\delta$ (which actually hold for any *convex* figure) [9]. So $\delta_k \asymp p_k$. Therefore, for $\delta_{k+1} \leq r \leq \delta_k$,

$$c''' r^{2-1/\alpha} \leq V(r),$$

for some positive number c'''. Both bounds are equivalent to

$$V(r) \asymp r^{2-1/\alpha},$$

implying that $\dim_B E = 1/\alpha$ $(1/2 < \alpha < 1)$.

In conclusion, the conditions

$$a_k \asymp k^{-2\alpha},\; a_k \asymp p_k^2 \tag{10}$$

seem to be as suitable for convex polygonal voids as for triangular voids or discs. In fact, an approximation argument would show that these conditions are suitable for general convex voids. The rationale for this proof is that the quotient p^2/a is the measure of shape for convex figures and its being bounded ensures that they do not degenerate in one-dimensional figures (segments), so that perimeter and area have the natural scaling behaviour.

2.4 Generalization to three-dimensional cut-out sets

We proceed to the generalization to three-dimensional cut-out sets with convex polyhedral voids. We need to generalize the geometrical properties that we have used from convex polygons to convex polyhedra.

Let A be a compact convex region that is to become a cut-out set. Falconer's formula (1) for the area of the r-neighbourhood of E can be generalized to three dimensions (ball-shaped voids):

$$V(r) = (ar + Hr^2 + \frac{4}{3}\pi r^3) + \sum_{i=1}^{k} \frac{4}{3}\pi\left(r_i^3 - (r_i - r)^3\right) + \sum_{i=k+1}^{\infty} \frac{4}{3}\pi r_i^3 \tag{11}$$

$$= (ar + Hr^2) + 4\pi r \sum_{i=1}^{k} r_i^2 - 4\pi r^2 \sum_{i=1}^{k} r_i + \frac{4}{3}\pi \sum_{i=k+1}^{\infty} r_i^3 + \frac{4}{3}\pi r^3(1+k), \tag{12}$$

where k is such that r is between the ball radii of indices k and $k+1$ ($r_{k+1} \leq r \leq r_k$). The first term of Eq. (11) is the volume of the layer surrounding A, given by Steiner's formula, where H is A's *linear measure* (mean curvature) [9]. The second term is the volume of the shells of width r inside the balls A_i, for $1 \leq i \leq k$, and the third term the area of the balls A_i for $i \geq k+1$. In the second expression, Eq. (12), the second term represents the sum of the areas of the balls A_i for $1 \leq i \leq k$ multiplied by r, while the third term represents the sum of their linear measures multiplied by r^2.

We assume that $r_k \asymp k^{-\alpha},\ 1/3 < \alpha < 1/2$, ensuring that the series of ball areas diverges while the series of their volumes converges. So, for $r_{k+1} \leq r \leq r_k$, we have

$$V(r) \asymp r + r^2 + r\sum_{i=1}^{k} i^{-2\alpha} + \sum_{i=k+1}^{\infty} i^{-3\alpha} + r^3 k \asymp r\, k^{1-2\alpha} + k^{1-3\alpha} + r^3 k \asymp r^{3-1/\alpha}, \tag{13}$$

and

$$\dim_B E = 3 - \lim_{r\to 0} \frac{\log V(r)}{\log r} = \frac{1}{\alpha} > 2.$$

Again, a cut-out set must have dimension larger than $d-1$ (the topological dimension of the boundaries of voids).

For tetrahedral voids, the r-neighbourhood of the boundary of a tetrahedron is a layer of thikness r and its volume is (in analogy with Eq. (4))

$$v - v' = ar - \frac{a^2 r^2}{3v} + \frac{a^3 r^3}{27 v^2}\,. \tag{14}$$

Here a and v are the area surface and volume of the tetrahedral void, respectively, and v' is the volume of the smaller homothetical tetrahedron. The factor $a^3/(27v^2)$ only depends on the angles of the tetrahedron and is a measure of its shape. The maximal r, such that v' vanishes, is $r = \rho = 3v/a$, that is, the radius of the inscribed sphere. The appropriate relations for tetrahedral voids are

$$\rho_k \asymp k^{-\alpha},\ a_k \asymp \rho_k^2,\ v_k \asymp \rho_k^3\,.$$

So the relation between surface area and volume to be extended to general convex polyhedra is $v_k \asymp a_k^{3/2}$. The lower bound to $a^{3/2}/v$ is again universal, according to a three-dimensional

isoperimetric inequality, and corresponds to a ball (for tetrahedra, to the regular tetrahedron). The upper bound forbids again small angles that give rise to flattened or spiky tetrahedra. Note that a tetrahedron can degenerate either into a two-dimensional or a one-dimensional figure (triangle or segment, respectively), but the former is more generic.

In the case of convex polyhedral voids, in analogy with two dimensions, a lower bound to $V(r)$ is provided by the sum of the volumes of polyhedra fully included in the r-neighbourhood and an upper bound is given by the three-dimensional version of Eq. (8). For the lower bound to $V(r)$, to determine when a void polyhedron is fully included in the r-neighbourhood of E, we use again the criterium given by the diameter δ of the polyhedra, namely, $r \geq \delta$. So, if we order the void polyhedra by their diameter and k is such that $\delta_{k+1} \leq r \leq \delta_k$,

$$\sum_{i=k+1}^{\infty} v_i \leq V(r). \tag{15}$$

For the upper bound to $V(r)$ we need a condition that ensures that a given polyhedron is not included in the r-neighbourhood of E. The volume of the layer of thikness r inside a polyhedron A_i is smaller than $a_i\, r$. We have (in analogy with inequality (8) for polygons):

$$V(r) \leq ar + Hr^2 + r\sum_{i=1}^{k} a_i + \sum_{i=k+1}^{\infty} v_i + \frac{4}{3}\pi r^3 \tag{16}$$

where the void polyhedra are ordered by their value of v/a and k is such that

$$\frac{v_{k+1}}{a_{k+1}} \leq r \leq \frac{v_k}{a_k}\,.$$

Assuming that

$$v_k \asymp k^{-3\alpha},\ v_k \asymp a_k^{3/2}\,, \tag{17}$$

it follows that

$$V(r) \leq c\, r^{3-1/\alpha},$$

for some positive number c, and $r < 1$.

For the lower bound to $V(r)$, we may relate the diameter of a convex polyhedron with its linear measure H. Like in the two-dimensional case, there are both lower and upper bounds to their ratio: $\delta < H/\pi$ and $H \leq 2\pi\delta$, which hold for any *convex* body [9]. H is *independent* of v and a, but the relations (17) imply $H_k \asymp k^{-\alpha}$ nonetheless. This follows from the two fundamental three-dimensional isoperimetric inequalities: $a^2 \geq 3vH,\ H^2 \geq 4\pi a$ [9]. Therefore, for $\delta_{k+1} \leq r \leq \delta_k$,

$$c' r^{3-1/\alpha} \leq V(r),$$

for some positive number c'. Both upper and lower-bound inequalities are equivalent to

$$V(r) \asymp r^{3-1/\alpha},$$

so $\dim_B E = 1/\alpha$ $(1/3 < \alpha < 1/2)$.

In conclusion, the conditions (17) are suitable for convex polyhedral voids and, furthermore, an approximation argument would show that they are suitable for general convex voids. Note that our proof relies on the sufficiency of the upper bound to the quotient a^3/v^2 (as a measure of shape) for preventing degeneracy into lower dimension. This holds regardless of the actual existence of two independent measures of shape of three-dimensional convex bodies.

3 Large scale distribution of matter and cosmic voids

The large scale distribution of matter in cosmology is produced by the gravitational instability of primordial small fluctuations in a homogenous Friedman-Robertson-Walker universe [13, 14]. The dynamics of structure formation is very nonlinear and, therefore, difficult to study with analytic methods. However, this dynamics is scale invariant, at least within some range of scales, due to the scale invariance of gravity. In consequence, it is natural that a scale invariant distribution of matter develops, with fractal geometry. Indeed, the fractal geometry of the distribution of galaxies has been studied for years [5], and the study of galaxy clustering actually stimulated the development of fractal geometry [1].

Mandelbrot considered in his book [1] the presence of voids in the distribution of galaxies but, according to the observational situation at the time of writing it, favored small voids and, actually, introduced the concept of *lacunarity* to account for this feature. The observation of large voids in the distribution of galaxies is more recent. Surprisingly, the cosmological literature on voids [6] hardly treats their fractal properties. Trying to fill this gap, we began a program to adapt the algorithmic studies of cosmic voids to general fractals and, viceversa, to discern fractal features in cosmic voids [3, 4]. In particular, we proposed to employ standard rank-ordering techniques and test the already compiled catalogues of cosmic voids for Zipf's law. In Ref. [4], we devised a void-finding algorithm based on discrete geometry methods, namely, Delaunay and Voronoi tessellations, that produces a sequence of voids of polyhedral shape (in $d = 3$).

The concepts of discrete stochastic geometry had been introduced in cosmology before by Rien van de Weygaert and collaborators, with different purposes (a comprehensive reference is [12]). One of these purposes was actually the construction of a model of large-scale structure formation based on Voronoi tessellations. In this model, Voronoi cells represent void regions while the matter is concentrated in their walls. The cell centers represent void germs (contrary to their role in our void-finding algorithm, in which they correspond to matter particles).

It is pertinent here to mention that there is a successful model of large-scale structure formation that produces walls as first structures, namely, the adhesion model [14] (the walls are called "pancakes" in this context). The full structure produced by this model is a self-similar pattern of interlocking walls that has been dubbed the "cosmic web" (or the "cosmic foam") [12]. As matter concentrate in the walls, there appear depleted regions,

that is, voids. On account of the self-similarity, the voids form a *hierarchy*, akin to the void distribution given by Zipf's law. Most studies of the adhesion model focus on the distribution of matter and its evolution, but it has been argued that it makes more sense to focus on the evolution of underdense regions. The rationale for this viewpoint is the "bubble theorem" [12]: the evolution of an underdense region is such that its initial slight asphericity decreases, contrary to the evolution of an overdense region. Therefore, the evolution of underdense regions is essentially decribed by the expansion of ellipsoidal voids, which become more spherical until they collapse with other voids, forming walls.

3.1 The Voronoi foam model

This model was proposed by Icke and Van de Weygaert [12] and follows the idea of expansion of ellipsoidal voids. It is based on a set of points that are initially the peaks of the gravitational potential, where the matter is underdense. They become the "expansion centres" of matter flowing outwards with uniform velocity. Voids form in this manner. Furthermore, when the flow from one void encounters the flow from an adjacent one, a wall forms half-way between their centres. The resulting distribution is a set of Voronoi cells, that is, a "Voronoi foam". These Voronoi cells are convex polyhedra. Moreover, if we assume that they form a self-similar pattern, then these patterns are particular cases of cut-out sets with convex polyhedral voids.

Unfortunately, the Voronoi foam model, such as has been formulated, is not sufficient to deduce the similarity properties of the cell pattern. If we understand the "expansion centres" as the set of relative maxima of the gravitational potential, we have to consider that this set is *dense* for an initial random Gaussian distribution. So hardly any void expansion seems possible, unless one selects a subset of "expansion centres", according to some principle, which will determine the final cell pattern. Of course, an obvious condition for a self-similar cell pattern is that the cell sizes fulfill Zipf's law, namely, the rank-ordering of their volumes satisfies $v_n \asymp n^{-e}$ for some e. To ensure that the cut-out set defined in this manner be a fractal, in the sense of having a box dimension strictly between two and three, we have to demand $1 < e < 3/2$ and the condition that forbids the appearance of *degenerate* (quasi-planar) shapes. Provided that the scale-invariant dynamics implies Zipf's law for the void cells, with $1 < e < 3/2$, the "bubble theorem" must imply that they are non-degenerate.

4 Discussion

We have proved the relation between the box dimension of a cut-out set E and the exponent of the corresponding Zipf law, for convex voids, in particular, for convex polygonal voids in $d = 2$ and convex polyhedral voids in $d = 3$. The particular forms of Zipf's law for voids in $d = 2$ and $d = 3$ are Eqs. (10) and (17), respectively. So we have extended Falconer's results in $d = 1$ and $d = 2$ (further extension to convex voids in any dimension d seems straightforward). If the Zipf law exponent is e ($= d\,\alpha$), the relation is $\dim_B E = d/e$.

Sufficient conditions for this relation to hold is that $1 < e < d/(d-1)$ and, in addition, the exclusion of *degenerate* void shapes. We expect that the case with more physical applications is $d = 3$. In particular, we have explored the application of our result to the large scale distribution of matter in cosmology.

It is useful to make a few remarks on the box-dimension formula. We have emphasized that cut-out sets must have topological codimension one, that is, topological dimension $d-1$. This is the reason why $e < d/(d-1)$, which implies $\dim_B E = d/e > d-1$, according to the known order of box and topological dimensions: if $e > d/(d-1)$, the $d-1$-measure of the boundaries of voids converges and $\dim_B E = d-1$. On the other hand, the order of box, Hausdorff-Besicovitch and topological dimensions is $\dim_T E \leq \dim_H E \leq \dim_B E$, so $e > 1$ implies that $\dim_H E < d$. However, we cannot determine if the inequality $\dim_T E \leq \dim_H E$ is strict, so we cannot tell if E is in fact a fractal (according to the usual definition). As a one-dimensional example, consider the "convergent sequence sets" $E^{(p)} = \{0, 1, 2^{-p}, 3^{-p}, 4^{-p}, \ldots\}$, $p > 0$, with $\dim_B E^{(p)} = 1/(p+1)$ [2]: they have all but one of their points isolated and $\dim_H E^{(p)} = 0$. Let us recall that the box dimension only depends on the size of the voids and not on their arrangement, but the Hausdorff-Besicovitch dimension can be altered by a rearrangement of the voids.

The mathematical requirement of having topological codimension one can be difficult to test in point sets obtained from physical observations, which are necessarily finite. In fact, the definition of void itself becomes uncertain in a finite set of points. This is why one must resort to void-finding algorithms. These algorithms may have free parameters, giving rise to different sets of voids according to their value. We noted in Ref. [4] the possibility of "percolation of voids", which must be avoided, by selecting parameter values that produce small convex voids. In this way, we expect, when the number of sampling points of a cut-out fractal set grows, that the set of voids found approaches the real set of voids. However, questions of convergence are difficult to treat in a rigorous way. If a fractal set E has $\dim_B E > d-1$, but has topological codimension larger than one, the cut-out set defined by the set of voids found by some algorithm is a fractal set $E' \supset E$ with the same box dimension $\dim_B E' = \dim_B E$, *and* with topological codimension one. Naturally, the boundary of the voids includes $E' - E$, with $\dim_H(E' - E) \leq \dim_B(E' - E) \leq d-1$, so it is negligible with respect to E. We have mentioned in section 2.1 an example belonging to Cantor-like fractals.

Regarding the application of our results to the large scale structure of matter, we can rely on both observational and theoretical results to support the existence of voids and scale invariance. The observations refer to the galaxy positions, rather than to the full dark matter distribution. For the moment, in spite of the presence of voids in the galaxy distribution, it is a moot point whether or not the galaxies are distributed along walls (even though the expression "wall galaxy" is in use, especially in the cosmological literature about voids). It is even more uncertain whether or not the voids satisfy Zipf's law [3]. The application of various void-finders to galaxy catalogues yields results that are not necessarily consistent. In contrast, the available theories of non-linear gravitational clustering support the existence and scaling of voids; in particular, the adhesion model leads to the formation of a self-similar "cosmic foam", as commented in Sect. 3. In this

context, the relation obtained here between the Zipf law for voids and box and Hausdorff-Besicovitch dimensions will surely be helpful.

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